Int. J. Heat Mass Transfer. Vol. 19, p. 1341. Pergamon Press 1976. Printed in Great Britain

REJOINDER

(*Received* 21 *February* 1976)

IN THE comments to my paper [5] three separate questions were touched upon. I am here giving an answer in the order proposed by the authors of the remarks. I should like to thank them very much for turning my attention to a very interesting paper [1], which I had not known.

In my case the paper [2] played an inspiring role, in which the expression describing the part of the functional corresponding to the elliptical operator in the form:

$$\operatorname{div}[\lambda(T)\operatorname{grad} T]$$

was presented.

The main idea of my paper [5] was presented at the conference in Zakopane in 1973 and published as an abstract in [3]. On account of the selection and preparation the lecture was declared earlier.

It is likely, that in the expression (2) of the author's comments some mistake has occurred, namely equation (2) according to their notation in [1] has the form:

$$\delta_t\left\{\int_V (\tfrac{1}{2}k^2(\operatorname{grad} T)^2 + \Gamma\cdot \dot{T})\,\mathrm{d}V + \int_{A_j} K\cdot n_i\cdot J_i^*\cdot T\cdot \mathrm{d}A + \int_{A_i} \dot{h}_s\cdot R\cdot \mathrm{d}A\right\} = 0$$

which corresponds in my notation to:

$$\delta_t\left\{\int_V \left[\frac{\lambda^2}{2}(T)(\operatorname{grad} T)^2 + \left(\frac{\partial T}{\partial \tau}\right)^* \int_{T_0}^{T} C(\Theta)\cdot\lambda(\Theta)\cdot \mathrm{d}\Theta\right]\mathrm{d}V + \int_{F_2} q\left[\int_{T_0}^{T}\lambda(\Theta)\,\mathrm{d}\Theta\right]T\cdot \mathrm{d}A + \underline{\alpha\int_{F_3}\left[\int_{T_0}^{T}\lambda(\Theta)(\Theta-\Theta_a)\,\mathrm{d}\Theta\right]\mathrm{d}A}\right\} = 0.$$

The part of the expression, which differs from that in their comments, was underlined. The expression connected with the heat sources and more general boundary conditions as well as the expression describing initial condition, were earlier used in my papers as for example [4].

In my earlier papers I applied the Laplace transformation in order to change the type of the differential equations. The application of the rules of variational calculus to the transformed differential equation and subsequent solution of this problem by means of approximate methods leads to the expression for $Z|_{t=0}$. The application of direct variational method, for example in the Kantorovich approach, in conjunction with the initial condition in the form:

$$Z\Big|_{t=0} = \tfrac{1}{2}\int_V (T-T_p)^2\cdot \mathrm{d}V$$

enables us to acquire the results as in the case of the Laplace transform.

In paper [1] the boundary conditions are discussed in detail, yet the initial condition was completely omitted. Would the initial condition really play the minor role in confrontation with the boundary conditions?

I should like to bring to the authors' attention that making use of the functional form of [5], we can not only make the initial condition but also independently satisfy the boundary conditions.

The next remark is connected with the expression: $\partial T/\partial\tau = F(x,y,z,\tau)$. The fact that $\partial T/\partial \tau$ was assumed to be the function of position and time does not mean that this function is known and prescribed.

As far as the functional presented by the authors is concerned, I would like to mention that it does not assume the stationary value, but depends on it. In order to make the expression presented in [1] stationary, one has to integrate this expression over the time. Because of this, I stopped dealing with the restricted variation. In the case of the functional given in [5] we can say that it assumes the stationary value. The Legendre conditions are connected with the second variation, whereby

$$\frac{\partial}{\partial T_\xi}\left(\frac{\partial L}{\partial T_\xi}\right) = \lambda^2 > 0$$

implies $\delta^2 J > 0$. The latter condition is necessary but, of course, not sufficient.

The Kantorovich method is based upon the assumption that the temperature can be approximated by the following expression:

$$T_n = \sum_0^n A_i(\tau)\cdot\varphi_i(\xi,\eta,\zeta)\quad [i = 0,1,2,\ldots,n].$$

The function $\varphi_i(\xi,\eta,\zeta)$ is assumed to be known, where A_i are undetermined functions of single variable τ. To approximate the solution, we use the expression for the first variation:

$$\int_0^t\left\{\int_V\left\{\frac{\partial T_n}{\partial\tau}\cdot C(T_n) - \operatorname{div}[\lambda(T_n)\operatorname{grad} T_n] - W(T_n)\right\}\lambda(T_n)\cdot\varphi_i\cdot \mathrm{d}V\right\}A_i\,\mathrm{d}\tau = 0\quad [i = 0,1,2,\ldots,n]$$

where, for clarity, we have omitted the expressions connected with the boundary and initial conditions.

From the idea of Kantorovich method it follows that the expression in brackets is equal to zero:

$$\int_V\left\{\frac{\partial T_n}{\partial \tau}C(T_n) - \operatorname{div}[\lambda(T_n)\operatorname{grad} T_n] - W(T_n)\right\}\lambda(T_n)\cdot\varphi_i\cdot \mathrm{d}V = 0.$$

The application of Kantorovich method finally leads to a set of ordinary differential equations. The equations (34)–(41) derived in work [5] was obtained on the base of this method.

It is surprising that the authors, being convinced of the superiority of Ritz–Rayleigh method, themselves proposed the Kantorovich method (see *Concluding Remarks* of [1]).

B. KRAJEWSKI

Department of Reactor Engineering
Institute of Nuclear Research
Swierk near Warsaw, Poland

REFERENCES

1. G. Lebon and J. Lambermont, Some variational principles pertaining to non-stationary heat conduction and coupled thermoelasticity, *Acta Mech.* **15**, 121–139 (1972).
2. Dj. Djukic and B. Vujanovic, On a new variational principle of Hamiltonian type for classical field theory, *Z. Angew. Math. Mech.* **51**(8), 611–616 (1971).
3. B. Krajewski, Direct variational method applied to nonlinear heat flow problems, 15th Polish Solid Mechanics Conference, Abstracts, Zakopane (3–12 September 1973).
4. B. Krajewski, Application of variational methods to problems of unsteady heat flow, *Arch. Mech. Stosow.* **5**, 20 (1968).
5. B. Krajewski, On a direct variational method for nonlinear heat transfer, *Int. J. Heat Mass Transfer* **18**, 495–502 (1975).